Problem 2.13

Consider a mass m constrained to move on the x axis and subject to a net force $F = -kx$ where k is a positive constant. The mass is released from rest at $x = x_o$ at time $t = 0$. Use the result (2.85) in Problem 2.12 to find the mass's speed as a function of x; that is, $dx/dt = g(x)$ for some function $g(x)$. Separate this as $dx/g(x) = dt$ and integrate from time 0 to t to find x as a function of t. (You may recognize this as one way—not the easiest—to solve the simple harmonic oscillator.)

Solution

Consider a mass constrained to move on the x-axis. By Newton's second law,

$$F = ma.$$

If the net force is $F = -kx$, then

$$\begin{aligned} -kx &= ma \\ &= m\frac{dv}{dt} \\ &= m\frac{dv}{dx}\frac{dx}{dt} \\ &= m\frac{dv}{dx}v \\ &= \frac{m}{2}\left(2v\frac{dv}{dx}\right) \\ &= \frac{m}{2}\frac{d}{dx}(v^2). \end{aligned}$$

Divide both sides by $m/2$.

$$-\frac{2k}{m}x = \frac{d}{dx}(v^2)$$

Integrate both sides from x_o to x.

$$\begin{aligned} \int_{x_o}^{x} -\frac{2k}{m}x'\,dx' &= \int_{x_o}^{x} \frac{d}{dx'}(v^2)\,dx' \\ -\frac{2k}{m}\frac{x'^2}{2}\bigg|_{x_o}^{x} &= v^2\bigg|_{x_o}^{x} \\ -\frac{2k}{m}\left(\frac{x^2}{2} - \frac{x_o^2}{2}\right) &= [v(x)]^2 - [v(x_o)]^2 \end{aligned}$$

Since the mass is released from rest, the initial velocity is zero: $v(x_o) = 0$.

$$v^2 = \frac{k}{m}(x_o^2 - x^2)$$

Take the square root of both sides to get v.

$$v(x) = \pm\sqrt{\frac{k}{m}(x_\text{o}^2 - x^2)}$$

The minus sign is chosen when the mass is moving to the left (such as initially if $x_\text{o} > 0$), and the positive sign is chosen when the mass is moving to the right (such as initially if $x_\text{o} < 0$). Replace the velocity with dx/dt to get an equation for the position.

$$\frac{dx}{dt} = \pm\sqrt{\frac{k}{m}(x_\text{o}^2 - x^2)}$$

$$= \pm\sqrt{\frac{k}{m}}\sqrt{x_\text{o}^2 - x^2}$$

Solve for x by separating variables

$$\frac{dx}{\sqrt{x_\text{o}^2 - x^2}} = \pm\sqrt{\frac{k}{m}}\,dt$$

and integrating both sides.

$$\int \frac{dx}{\sqrt{x_\text{o}^2 - x^2}} = \int \pm\sqrt{\frac{k}{m}}\,dt$$

$$\int^x \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}} = \pm\sqrt{\frac{k}{m}}\,t + C$$

Use the fact that $x = x_\text{o}$ at time $t = 0$ to determine C.

$$\int^{x_\text{o}} \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}} = C$$

Consequently,

$$\int^x \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}} = \pm\sqrt{\frac{k}{m}}\,t + \int^{x_\text{o}} \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}}$$

$$\int^x \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}} - \int^{x_\text{o}} \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}} = \pm\sqrt{\frac{k}{m}}\,t$$

$$\int^x \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}} + \int_{x_\text{o}} \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}} = \pm\sqrt{\frac{k}{m}}\,t$$

$$\int_{x_\text{o}}^x \frac{dx'}{\sqrt{x_\text{o}^2 - x'^2}} = \pm\sqrt{\frac{k}{m}}\,t$$

For the integral on the left, make a trigonometric substitution.

$$x' = x_\text{o}\cos\theta'$$

$$dx' = -x_\text{o}\sin\theta'\,d\theta'$$

As a result,

$$\pm\sqrt{\frac{k}{m}}\,t = \int_0^{\cos^{-1}\left(\frac{x}{x_o}\right)} \frac{-x_o\sin\theta'\,d\theta'}{\sqrt{x_o^2-(x_o\cos\theta')^2}}$$

$$= -\frac{x_o}{|x_o|}\int_0^{\cos^{-1}\left(\frac{x}{x_o}\right)} \frac{\sin\theta'\,d\theta'}{\sqrt{1-\cos^2\theta'}}$$

$$= -\operatorname{sgn} x_o \int_0^{\cos^{-1}\left(\frac{x}{x_o}\right)} \frac{\sin\theta'\,d\theta'}{\sin\theta'}$$

$$= -\operatorname{sgn} x_o \int_0^{\cos^{-1}\left(\frac{x}{x_o}\right)} d\theta'$$

$$= -\operatorname{sgn} x_o \cos^{-1}\left(\frac{x}{x_o}\right).$$

Solve for the inverse cosine function.

$$\cos^{-1}\left(\frac{x}{x_o}\right) = \frac{\mp 1}{\operatorname{sgn} x_o}\sqrt{\frac{k}{m}}\,t$$

This means that

$$\frac{x}{x_o} = \cos\left(\frac{\mp 1}{\operatorname{sgn} x_o}\sqrt{\frac{k}{m}}\,t\right)$$

$$= \cos\left(\sqrt{\frac{k}{m}}\,t\right).$$

Therefore,

$$x(t) = x_o\cos\left(\sqrt{\frac{k}{m}}\,t\right).$$